Sec 5.1 2^{nd} order linear ODEs

Most general form:

$$y'' + A(x)y' + B(x)y = \underbrace{F(x)}$$

if $F(x)=0$ always, eqn is <u>homogeneous</u>.

Most common form for us: constant coeffs:

$$\boxed{y'' + a_1 y' + a_0 y = 0} \text{ (homog.)}$$

<u>Big idea:</u> solution functions can be built from "basis functions"

<u>Ex</u>: $y''=0 \longrightarrow$ solutions are $a\boxed{x} + b\cdot\boxed{1}$

solutions are lin comb. of $\{x, 1\}$

const. func

so $\{x, 1\}$ make a basis for "solution space for $\{y''=0\}$"

$$\left(\text{idea: } \quad x \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad 1 \to \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad y = ax+b \longleftrightarrow \begin{bmatrix} a \\ b \end{bmatrix} \right)$$

<u>Ex</u> general solutions for $\{y''+y=0\}$ are lin comb. $y(x) = c_1 \cos(x) + c_2 \sin(x)$

Main thm/Idea: General solutions for $y'' + A(x)y' + B(x)y = 0$ are lin. comb. $y(x) = c_1 \boxed{y_1} + c_2 \boxed{y_2}$, where y_1 and y_2 are linearly indep func.

* y_1, y_2 lin depen only if $y_1 = k y_2$
(const)

How to find solutions: (by example)

- If ODE is $y'' + ay' + by = 0$, use characteristic equation $\boxed{r^2 + ar + b = 0}$
- Find roots r_1, r_2 of $r^2 + ar + b = 0$

- **Case 1**: $r_1 \neq r_2$ general solution of ODE is:
$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Ex: $y'' - 5y' + 6y = 0$

char. eq. $r^2 - 5r + 6 = 0$

$$(r-3)(r-2) = 0$$

$$\Rightarrow r = 3, \ r = 2 \quad (r_1 \neq r_2)$$

gen sol is $\boxed{y(x) = c_1 e^{2x} + c_2 e^{3x}}$

Ex: If want particular solution such that $\{ y'' - 5y' + 6 = 0, \ y(0) = 3, \ y'(0) = -1 \}$

$$y(x) = c_1 e^{2x} + c_2 e^{3x}$$
$$y'(x) = 2c_1 e^{2x} + 3c_2 e^{3x}$$

$$\left.\begin{array}{l} y(0) = c_1 + c_2 = 3 \\ y'(0) = 2c_1 + 3c_2 = -1 \end{array}\right\} \quad \begin{array}{l} c_2 = -7 \\ c_1 = 10 \end{array}$$

particular solution

$$\underline{y(x) = 10e^{2x} - 7e^{3x}}$$

- <u>Case $r_1 = r_2 =$ "r"</u>

Can't use e^{rx} twice (need 2 lin indep funcs.)

$$\boxed{\text{gen sol is } \quad y(x) = c_1 \boxed{e^{rx}} + c_2 \boxed{xe^{rx}}}$$

- Ex gen sol $\quad y'' + 2y' + y = 0$

ch eq. $\quad r^2 + 2r + 1 = 0$

$$(r+1)^2 = 0 \Rightarrow r=-1, r=-1$$
(repeat)

gen sol is $\quad y(x) = c_1 e^{-x} + c_2 x e^{-x}$

What is indepence?

<u>DEF</u> funcs f, g are lin <u>depen</u> if
lin comb $c_1 f + c_2 g = 0$ (const 0)
<u>is</u> possible with $(c_1, c_2) \neq (0,0)$.
iff <u>only</u> possible with $(c_1, c_2) = (0,0)$

<u>test:</u>

$$c_1 f + c_2 g = 0$$

$$\Rightarrow \quad c_1 f' + c_2 g' = 0$$

$$\Rightarrow \quad \begin{bmatrix} f & g \\ f' & g' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

functions are indep/depen based on

$$W(f,g) \overset{\text{DEF}}{=} \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g$$

"Wronskian"

Ex: $W(e^{2x}, e^{-x}) = \begin{vmatrix} e^{2x} & e^{-x} \\ 2e^{2x} & -e^{-x} \end{vmatrix}$

$$= -e^{x} - (2e^{x})$$

$$= -3e^{x} \underline{\neq 0},$$

so e^{2x}, e^{-x} <u>lin indep</u>.